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Extremal bounds for pattern avoidance in multidimensional 0-1 matrices. (English. English summary)

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The authors study an analogue of the Turán function for multidimensional 0-1 matrices. Let $d \geq 2$. Say that a d-dimensional 0-1 matrix A *contains* (resp., *strictly contains*) a d-dimensional 0-1 matrix P if there is a submatrix of A that equals P after changing some (resp., at least one) of the 1-entries of A to 0-entries; if A does not contain P, then A is said to be P-*free*. For a collection $\mathcal{P}$ of d-dimensional 0-1 matrices, A is said to be $\mathcal{P}$-*free* if it is P-free for all $P \in \mathcal{P}$. The extremal function $\mathrm{ex}(n, \mathcal{P}, d)$ is the maximum number of 1-entries in any d-dimensional $n \times \cdots \times n$ 0-1 matrix A that is $\mathcal{P}$-free. Also, define the *weight* of A to be the number of 1-entries in A.

It is easy to see that if P has at least two 1-entries, then $\mathrm{ex}(n, P, d) = \Omega(n^{d-1})$. In the first half of the paper, the authors investigate the d-dimensional analogue of a problem posed by Z. Füredi and P. Hajnal [Discrete Math. **103** (1992), no. 3, 233–251; MR1171777] for the case $d = 2$, namely to characterize the d-dimensional matrices P for which $\mathrm{ex}(n, P, d) = O(n^{d-1})$. For this, one defines a *minimally non-$O(n^{d-1})$ d-dimensional 0-1 matrix* P to be one for which $\mathrm{ex}(n, P, d) = \omega(n^{d-1})$ but $\mathrm{ex}(n, P', d) = O(n^{d-1})$ for every d-dimensional matrix P' that P strictly contains. It is easy to see that given a minimally non-$O(1)$ (i.e., *minimally nonlinear*) matrix of size $j \times k$, one can generate minimally non-$O(n^{d-1})$ d-dimensional 0-1 matrices of size $j \times k \times 1 \times \cdots \times 1$, for each $d > 2$. The authors show that there are infinitely many examples where each dimension has length greater than 1, and they construct several specific examples having weight 4 in dimension 3. They prove upper bounds on the weight and on the number of minimally non-$O(n^{d-1})$ d-dimensional 0-1 matrices in terms of the lengths of the dimensions. The authors also identify a parametrized family $\mathcal{J}_{n_0,d} \cup \mathcal{D}_{n_0,d}$ of d-dimensional 0-1 matrices, each of dimension $n_0 \times \cdots \times n_0$ and weight n_0, such that the following holds: for any family $\mathcal{P}$ of d-dimensional 0-1 matrices, if there exists $n_0 \in \mathbb{N}$ such that no element of $\mathcal{J}_{n_0,d} \cup \mathcal{D}_{n_0,d}$ avoids all $P \in \mathcal{P}$, then $\mathrm{ex}(n, \mathcal{P}, d) = O(1)$, else $\mathrm{ex}(n, \mathcal{P}, d) \geq n$. This generalizes a result of G. Tardos [J. Combin. Theory Ser. A **111** (2005), no. 2, 266–288; MR2156213] for the case $d = 2$. Lastly, they show that if $|\mathcal{P}| = k$, where $k < d$, then $\mathrm{ex}(n, \mathcal{P}, d)$ is either 0 or at least n^{d-k}. Moreover, for every choice of integers d, k, r such that $d, k > 0$ and $r \in \{0, 1, \ldots, d-1\}$, there is a family $\mathcal{P}_{d,k,r}$ of d-dimensional 0-1 matrices such that $\mathrm{ex}(n, \mathcal{P}_{d,k,r}, d) = kn^r$ for all sufficiently large n, and the smallest such family has size $d - r$.

In the second half of the paper, the authors study saturation properties of these matrices. Say that a matrix A is $\mathcal{P}$-*saturated* if it is P-free for all $P \in \mathcal{P}$, but changing any 0-entry of A into a 1-entry results in a matrix that contains some $P \in \mathcal{P}$. Say that a matrix A is $\mathcal{P}$-*semisaturated* if changing any 0-entry of A into a 1-entry results in a matrix that has a new copy of some $P \in \mathcal{P}$. The *saturation function* $\mathrm{sat}(n, \mathcal{P}, d)$ (resp., *semisaturation function* $\mathrm{ssat}(n, \mathcal{P}, d)$) is the minimum number of 1-entries in any d-dimensional $n \times \cdots \times n$ 0-1 matrix which is $\mathcal{P}$-saturated (resp., $\mathcal{P}$-semisaturated). The authors show that for any family $\mathcal{P}$ of d-dimensional 0-1 matrices, there is an integer $k \in \{0, 1, \ldots, d-1\}$ such that $\mathrm{ssat}(n, \mathcal{P}) = \Theta(n^k)$, and they give necessary and sufficient conditions to find the integer k. Moreover, for every $d \geq 2$ and every integer $k \in \{0, 1, \ldots, d-1\}$, there is a family $\mathcal{P}$ of d-dimensional 0-1 matrices for which $\mathrm{ssat}(n, \mathcal{P}, d) = \Theta(n^k)$. Next, they show that for any family $\mathcal{P}$ of nonempty d-dimensional 0-1 matrices, $\mathrm{sat}(n, \mathcal{P}, d) = O(1)$ or $\Omega(n)$. This generalizes a result of R. Fulek and B. Keszegh [SIAM J. Discrete Math. **35** (2021), no. 3, 1964–1977; MR4305785] for the case $d = 2$. Lastly, just as for the extremal function, we have for each choice of d, k, r, where

$d, k > 0$ and $r \in \{0, 1, \ldots, d-1\}$, there is a family $\mathcal{P}_{d,k,r}$ of d-dimensional 0-1 matrices such that $\mathrm{sat}(n, \mathcal{P}_{d,k,r}, d) = kn^r$ for all sufficiently large n. Also, the gap between the saturation and extremal functions can be arbitrarily close to the maximum possible gap: for every $d \geq 2$ and $\varepsilon > 0$, there is a family $\mathcal{P}$ of d-dimensional 0-1 matrices such that $\mathrm{sat}(n, \mathcal{P}, d) = O(1)$ and $\mathrm{ex}(n, \mathcal{P}, d) = \Omega(n^{d-\varepsilon})$.

The authors ask whether the upper bounds obtained in the first half can be sharpened, and also whether the growth rates of the extremal functions for the 3-dimensional matrices constructed in the first half can be found up to optimal constants. The minimum possible size of the family $\mathcal{P}_{d,k,r}$ for which $\mathrm{sat}(n, \mathcal{P}_{d,k,r}, d) = kn^r$ is also not known, unlike the corresponding result for the extremal function. Lastly, the authors conjecture that for any family $\mathcal{P}$ of d-dimensional 0-1 matrices, there is an integer $r \in \{0, 1, \ldots, d-1\}$ such that $\mathrm{sat}(n, \mathcal{P}, d) = \Theta(n^r)$. *Brahadeesh Sankarnarayanan*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.